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LETTER TO THE EDITOR

Exact results for an O(n) model in two dimensions

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Abstract. As a step toward proving the generally held belief that the two-dimensional O(n) model exhibits no phase transition for $n \geqslant 3$, we consider a special O(n) model on the honeycomb lattice and establish a rigorous lower bound on n such that the free energy of the O(n) model is analytic for all n above this bound.

The O(n) model, first introduced by Stanley (1968) as a means of realising the spherical model by taking the $n \to \infty$ limit, has been a subject of increasing recent interest. The partition function of the O(n) model is

$$Z_n = \int \prod_{\langle ij \rangle} B(\boldsymbol{S}_i, \boldsymbol{S}_j) \prod_k \mathrm{d}\boldsymbol{S}_k \tag{1}$$

where $\boldsymbol{S}_i$ is a classical n-component vector of length $\sqrt{n}$ located at the ith site and $B(\boldsymbol{S}_i, \boldsymbol{S}_j)$ is the nearest-neighbour Boltzmann factor. We have for the *true* O(n) model

$$B(\boldsymbol{S}, \boldsymbol{S}') = \exp(K\boldsymbol{S} \cdot \boldsymbol{S}') \tag{2}$$

and we shall assume the normalisation $\int \mathrm{d}\boldsymbol{S} = 1$.

Several years ago, Domany *et al* (1981) established the following result for the honeycomb lattice. They showed that if the Boltzmann factor takes the special form

$$B(\boldsymbol{S}, \boldsymbol{S}') = 1 + x\boldsymbol{S} \cdot \boldsymbol{S}' \tag{3}$$

then the partition function (1) can be expanded into a graph-generating function

$$Z_n(x) = \sum_G x^b n^l. \tag{4}$$

Here the summation is taken over all graphs G (on the lattice) consisting of b bonds and l non-intersecting loops. It is clear that if we take (3) as the given definition of the Boltzmann factor, then the positivity of the Boltzmann factor requires x to be in the range

$$0 < x < 1/n. \tag{5}$$

Indeed, both the partition function (4) and the range (5) for x can be realised in some O(n) models. For instance, consider a *discrete* O(n) model and regard it as the (n, 2) model considered by Domany and Riedel (1979). In their notation and in the subspace of $\mathrm{e}^{C_2}(1+\mathrm{e}^{C_1}) = 2$, it is straightforward to verify that the O(n) partition function is given precisely by (4) with $x = n^{-1}\tanh(C_1/2)$ which is in the range (5). The O(n) partition function (4) also arises in the modelling of critical roughening of surfaces (Rys 1986) with x in the range

$$0 < x < 1. \tag{6}$$

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It appears that, in all physical applications (see also the discussions below) of the partition function (4), the range of x is always bounded from the above. This is a crucial fact we shall use in our analysis. In the ensuing discussions we shall use (4) as the definition of our $O(n)$ partition function with x bounded.

Using the analogue of a Coulomb gas, Nienhuis (1982, 1984) was able to determine the critical behaviour and the critical point of the partition function (4) for $-2 \leqslant n \leqslant 2$. For integral $n \geqslant 3$ for which the Nienhuis analysis does not apply, it is generally believed that the $O(n)$ model in two dimensions does not have a phase transition. But, to our knowledge, a rigorous proof of this statement has been lacking. In this letter we give some positive results on this unresolved problem.

Consider the $O(n)$ partition function (4) for the honeycomb lattice. First, we note, as in Nienhuis (1982, 1984), that there exists a one-to-one correspondence between directed polygonal configurations on the honeycomb lattice and ice-rule configurations on its surrounding (Kagomé) lattice. The six ice-rule vertex configurations are shown in figure 1 where, for the purpose of orientation, the hexagonal faces of the Kagomé lattice are identified by corners of 120°. The above-mentioned mapping of configurations is most easily seen by adopting the following convention, which avoids the introduction of the intermediate SOS model used by Nienhuis (1982, 1984). Draw a bond (on the honeycomb lattice) and direct it in the direction of the ice-rule polarisation for vertices of types (1) and (2). A typical global mapping between the configurations obtained in this fashion is shown in figure 2.

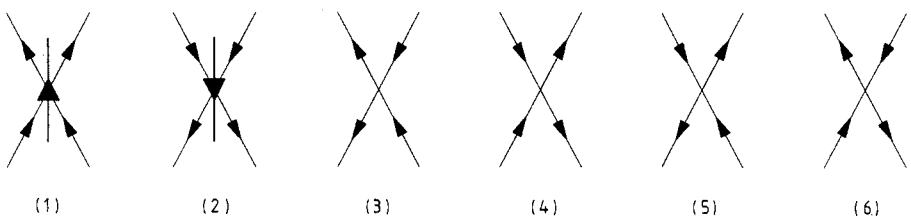

Figure 1. The six vertex configurations on the Kagomé lattice.

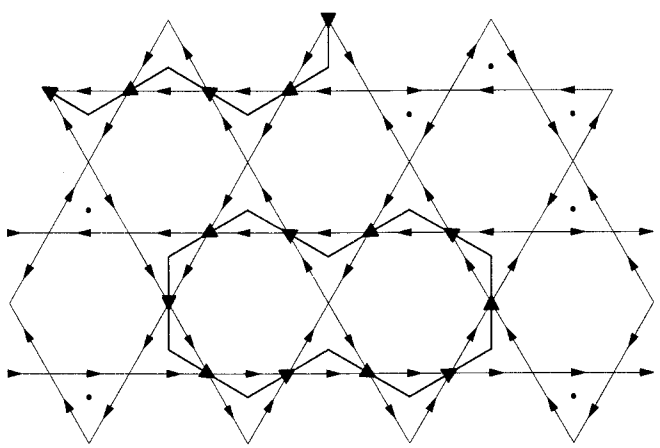

Figure 2. A mapping between a directed polygonal configuration on the honeycomb lattice and an ice-rule configuration on the Kagomé surrounding lattice.

Consider next the ice-rule model with vertex weights

$$\{\omega_1, \ldots, \omega_6\} = \{\tau, \tau, 1, 1, e^{-2\lambda}, e^{2\lambda}\}. \tag{7}$$

Then, following Nienhuis (1982, 1984), by associating factors $e^{-\lambda}$ and e^{λ} to left- and right-turning ice-rule arrows about corners of 60° and recollecting these factors globally, one obtains the elegant equivalence

$$Z_{6v}(\tau, \tau, 1, 1, e^{-2\lambda}, e^{2\lambda}) = (2 \cosh 3\lambda)^N Z_n(x) \tag{8}$$

where

$$n = 2 \cosh 6\lambda \qquad \tau = 2x \cosh 3\lambda \tag{9}$$

N is the number of sites of the honeycomb lattice and Z_{6v} is the partition function of the six-vertex ice-rule model with weights shown therein.

The Z_{6v} in (8) can be rewritten by taking out a common factor τ for each of the $3N/2$ vertices of the Kagomé lattice, resulting in

$$Z_{6v}(\tau, \tau, 1, 1, e^{-2\lambda}, e^{2\lambda}) = \tau^{3N/2} Z_{6v}(1, 1, x_r, x_r, c_r z, c_r z^{-1}) \tag{10}$$

with

$$x_r = c_r = \tau^{-1} \qquad z = e^{-2\lambda}. \tag{11}$$

The new Z_{6v} on the RHS of (10) is of precisely the form of Hintermann *et al* (1978) who considered the zeros of the ice-rule partition function. Using the result established in the appendix to Hintermann *et al* (1978), we arrive at the conclusion that, in a (generally complex) neighbourhood of real z and τ satisfying $\operatorname{Re} \lambda > 0$, the partition function $Z_{6v}(1, 1, x_r, x_r, c_r z, c_r z^{-1})$ is free of zeros in the region

$$|z| < [1 + (1 + |\tau|)^2]^{1/2} - 1 - |\tau|. \tag{12}$$

Combining (8)–(11), (12) then implies the analyticity of the free energy of the O(n) partition function (4) in the region

$$|2x \cosh 3\lambda| < \sinh 4(\operatorname{Re} \lambda) - 1. \tag{13}$$

This is our main result.

Clearly, the bound (13) is applicable only for $n > 2$ for which $\lambda > 0$. In O(n) models, where the variable x is restricted to a range $0 < x < x_0$, this yields a lower bound on n obtained from (13) by replacing x with x_0. Thus, this bound depends explicitly on the value of x_0. Particularly for $x_0 = n^{-1}$ and $x_0 = 1$, we find that the free energy is analytic, and consequently the O(n) model (4) exhibits no phase transition for

$$n > 6.025 \qquad \text{for } 0 < x < 1/n \tag{14}$$

and

$$n > 115.636 \qquad \text{for } 0 < x < 1. \tag{15}$$

These conclusions are consistent with the conjecture of the absence of a transition for $n \geqslant 3$.

Finally, we discuss the relevance of our results on the *true* O(n) model (2). We have already seen that the range (5), and hence the bound (14), is exact in a special

subspace of the discrete O(n) model. For the true O(n) model (2), however, we may regard (3), and hence (4), as some form of an approximation. One way to effect this approximation is to require identical value for the expectation value $\int \boldsymbol{S}\cdot\boldsymbol{S}'B(\boldsymbol{S},\boldsymbol{S}')\,\mathrm{d}\boldsymbol{S}\,\mathrm{d}\boldsymbol{S}'/\int B(\boldsymbol{S},\boldsymbol{S}')\,\mathrm{d}\boldsymbol{S}\,\mathrm{d}\boldsymbol{S}'$ evaluated using $B(\boldsymbol{S},\boldsymbol{S}')$ given by either (2) or (3). In this way one is led to

$$x=\frac{\int \boldsymbol{S}\cdot\boldsymbol{S}'\exp(K\boldsymbol{S}\cdot\boldsymbol{S}')\,\mathrm{d}\boldsymbol{S}\,\mathrm{d}\boldsymbol{S}'}{n\int\exp(K\boldsymbol{S}\cdot\boldsymbol{S}')\,\mathrm{d}\boldsymbol{S}\,\mathrm{d}\boldsymbol{S}'} \tag{16}$$

an expression relating x to the physical temperature K. The expression (16) is exact for $n=1$. It is also reasonably accurate in determining the critical x_c for other values of n. For $n=2$, e.g., using the Monte Carlo data of $K_c=1.085$ for the square lattice (Minnhagen and Nylen 1985), we obtain from (16) the critical $x_c=0.7238$. This is to be compared with the exact value of $x_c=1/\sqrt{2}=0.7071$ for the honeycomb lattice (Wu 1979, Domany *et al* 1981, Enting and Wu 1982). It should be noted that (for $K>0$) the value of x given by (16) lies in the range $0<x<1$ so that it is the bound (15) that applies in this approximation.

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References

Domany E, Mukamel D, Nienhuis B and Schwimmer A 1981 *Nucl. Phys.* B **190** [FS3] 279
Domany E and Riedel E K 1979 *Phys. Rev.* B **19** 5817
Enting I and Wu F Y 1982 *J. Stat. Phys.* **28** 353
Hintermann A, Kunz H and Wu F Y 1978 *J. Stat. Phys.* **19** 623
Minnhagen P and Nylen M 1985 *Phys. Rev.* B **31** 1693
Nienhuis B 1982 *Phys. Rev. Lett.* **49** 1062
—— 1984 *J. Stat. Phys.* **34** 731
Rys F S 1986 *Phys. Rev. Lett.* **56** 624
Stanley H E 1968 *Phys. Rev. Lett.* **20** 589
Wu F Y 1979 *J. Phys. C: Solid State Phys.* **10** L23